13 RULES *That Expire*

Overgeneralizing commonly accepted strategies, using imprecise vocabulary, and relying on tips and tricks that do not promote conceptual mathematical understanding can lead to misunderstanding later in students' math careers.

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Imagine the following scenario: A primary teacher presents to her students the following set of number sentences:

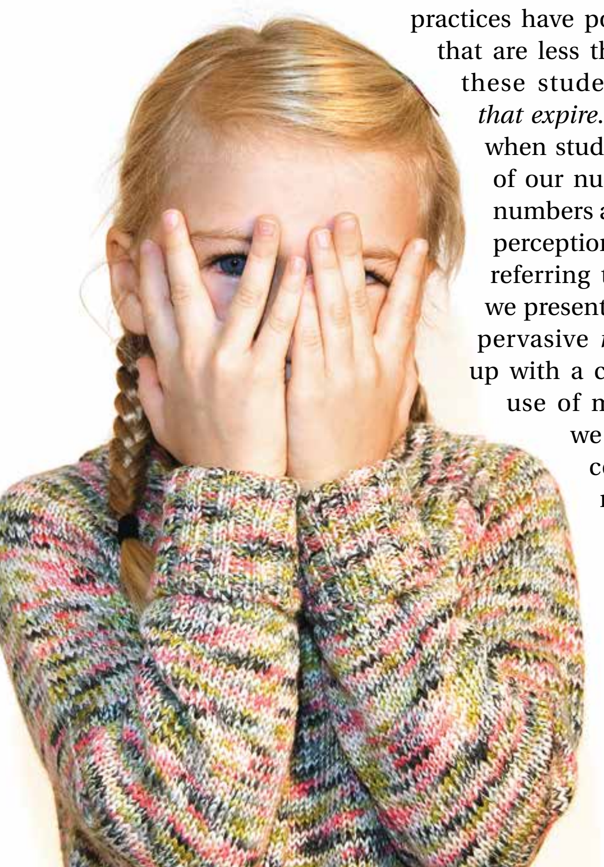
$$3 + 5 = \square$$
$$\square + 2 = 7$$
$$8 = \square + 3$$
$$2 + 4 = \square + 5.$$

Stop for a moment to think about which of these number sentences a student in your class would solve first or find easiest. What might they say about the others? In our work with young children, we have found that students feel comfortable solving the first equation because it "looks right" and students can interpret the equal sign as *find the answer*. However, students tend to hesitate at the remaining number sentences because they have yet to interpret and understand the equal sign as a symbol indicating a relationship between two quantities (or amounts) (Mann 2004).

In another scenario, an intermediate student is presented with the problem 43.5 × 10. Immediately, he responds, "That's easy; it is 43.50 because my teacher said that when you multiply any number times ten, you just add a zero at the end."

In both these situations, hints or repeated practices have pointed students in directions that are less than helpful. We suggest that these students are experiencing *rules that expire*. Many of these rules "expire" when students expand their knowledge of our number systems beyond whole numbers and are forced to change their perception of what can be included in referring to *a number*. In this article, we present what we believe are thirteen pervasive *rules that expire*. We follow up with a conversation about incorrect use of mathematical language, and we present alternatives to help counteract common student misunderstandings.

The Common Core State Standards (CCSS) for Mathematical Practice advocate for students to become problem solvers who can reason, apply, justify, and effectively use appropriate mathematics vocabulary to demonstrate their understanding of mathematics concepts (CCSSI 2010). This, in fact, is quite opposite of the classroom in which the teacher does most of the talking and students are encouraged to memorize facts, "tricks," and tips to make the mathematics "easy." The latter classroom can leave students with a collection of explicit, yet arbitrary, rules that do not link to reasoned judgment (Hersh 1997) but instead to learning without thought (Boaler 2008). The purpose of this article is to outline common rules and vocabulary that teachers share and elementary school students tend to overgeneralize—tips and tricks that do not promote conceptual understanding, rules that "expire" later in students' mathematics careers, or vocabulary that is not precise. As a whole, this article aligns to Standard of Mathematical Practice (SMP) 6: *Attend to precision*, which states that mathematically proficient students "…try to communicate precisely to others. …use clear definitions … and … carefully formulated explanations…" (CCSSI 2010, p. 7). Additionally, we emphasize two other mathematical practices: SMP 7: *Look for and make use of structure* when we take a look at properties of numbers; and SMP 2: *Reason abstractly and quantitatively* when we discuss rules about the meaning of the four operations.



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"Always" rules that are not so "always"

In this section, we point out rules that seem to hold true at the moment, given the content the student is learning. However, students later find that these rules are not always true; in fact, these rules "expire." Such experiences can be frustrating and, in students' minds, can further the notion that mathematics is a mysterious series of tricks and tips to memorize rather than big concepts that relate to one another. For each rule that expires, we do the following:

1. State the rule that teachers share with students.
2. Explain the rule.
3. Discuss how students inappropriately overgeneralize it.
4. Provide counterexamples, noting when the rule is not true.

5. State the “expiration date” or the point when the rule begins to fall apart for many learners. We give the expiration date in terms of grade levels as well as CCSSM content standards in which the rule no longer “always” works.

Thirteen rules that expire

1. When you multiply a number by ten, just add a zero to the end of the number.

This “rule” is often taught when students are learning to multiply a whole number times ten. However, this directive is not true when multiplying decimals (e.g., $0.25 \times 10 = 2.5$, not 0.250). Although this statement may reflect a regular pattern that students identify with whole numbers, it is not generalizable to other types of numbers. Expiration date: Grade 5 (5.NBT.2).

2. Use keywords to solve word problems.

This approach is often taught throughout the elementary grades for a variety of word problems. Using keywords often encourages students to strip numbers from the problem and use them to perform a computation outside of the problem context (Clement and Bernhard 2005). Unfortunately, many keywords are common English words that can be used in many different ways. Yet, a list of keywords is often given so that word problems can be translated into a symbolic, computational form. Students are sometimes told that if they see the word *altogether* in the problem, they should always add the given numbers. If they see *left* in the problem, they should always subtract the numbers. But reducing the meaning of an entire problem to a simple scan for key words has inherent challenges. For example, consider this problem:

> John had 14 marbles in his left pocket. He had 37 marbles in his right pocket. How many marbles did John have?

If students use keywords as suggested above, they will subtract without realizing that the problem context requires addition to solve. Keywords become particularly troublesome when students begin to explore multistep word problems, because they must decide which keywords work with which component of the problem. Keywords can be informative but must be used in conjunction with all other words in the problem to grasp the full meaning. Expiration date: Grade 3 (3.OA.8).

3. You cannot take a bigger number from a smaller number.

Students might hear this phrase as they first learn to subtract whole numbers. When students are restricted to only the set of whole numbers, subtracting a larger number from a smaller one results in a negative number, an integer that is not in the set of whole numbers, so this rule is true. Later, when students encounter application or word problems involving contexts that include integers, students learn that this “rule” is not true for all problems. For example, a grocery store manager keeps the temperature of the produce section at 4 degrees Celsius, but this is 22 degrees too hot for the frozen food section. What must the temperature be in the frozen food section? In this case, the answer is a negative number, ($4^{\circ} - 22^{\circ} = -18^{\circ}$). Expiration date: Grade 7 (7.NS.1).

4. Addition and multiplication make numbers bigger.

When students begin learning about the operations of addition and multiplication, they are often given this rule as a means to develop a generalization relative to operation sense. However, the rule has multiple counterexamples. Addition with zero does not create a sum larger than either addend. It is also untrue when adding two negative numbers (e.g., $-3 + -2 = -5$), because -5 is less than both addends. In the case of the equation below, the product is smaller than either factor.

$$\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$

This is also the case when one of the factors is a negative number and the other factor is positive, such as $-3 \times 8 = -24$. Expiration date: Grade 5 (5.NF.4 and 5.NBT.7) and again at Grade 7 (7.NS.1 and 7.NS.2).

5. Subtraction and division make numbers smaller.

This rule is commonly heard in grade 3: both subtraction and division will result in an answer that is smaller than at least one of the numbers in the computation. When numbers are positive whole numbers, decimals, or fractions, subtracting will result in a number that is smaller than at least one of the numbers involved in the computation. However, if the subtraction involves two negative numbers, students may notice a contradiction (e.g., $-5 - (-8) = 3$). In division, the rule is true if the numbers are positive whole numbers, for example:

$$8 \div 4 = 2 \text{ or } 4 \div 8 = \frac{1}{2}$$

However, if the numbers you are dividing are fractions, the quotient may be larger:

$$\frac{1}{4} \div \frac{2}{5} = \frac{5}{8}$$

This is also the case when dividing two negative factors: (e.g., $-9 \div -3 = 3$). Expiration dates: Grade 6 (6.NS.1) and again at Grade 7 (7.NS.1 and 7.NS.2c).

6. You always divide the larger number by the smaller number.

This rule may be true when students begin to learn their basic facts for whole-number division and the computations are not contextually based. But, for example, if the problem states that Kate has 2 cookies to divide among herself and two friends, then the portion for each person is $2 \div 3$. Similarly, it is possible to have a problem in which one number might be a fraction:

> Jayne has $\frac{1}{2}$ of a pizza and wants to share it with her brother. What portion of the whole pizza will each get?

In this case, the computation is as follows:

$$\frac{1}{2} \div 2 = \frac{1}{4}$$

Expiration date: Grade 5 (5.NF.3 and 5.NF.7).

7. Two negatives make a positive.

Typically taught when students learn about multiplication and division of integers, rule 7 is to help them determine the sign of the product or quotient. However, this rule does not always hold true for addition and subtraction of integers, such as in $-5 + (-3) = -8$. Expiration date: Grade 7 (7.NS.1).

8. Multiply everything inside the parentheses by the number outside the parentheses.

As students are developing the foundational skills linked to order of operations, they are often told to first perform multiplication on the numbers (terms) within the parentheses. This holds true only when the numbers or variables inside the parentheses are being added or subtracted, because the distributive property is being used, for example, $3(5 + 4) = 3 \times 5 + 3 \times 4$. The rule is untrue when multiplication or division occurs in the parentheses, for example, $2\ (4 \times 9) \neq 2 \times 4 \times 2 \times 9$. The 4 and the 9 are not two separate terms, because they are not separated by a plus or minus sign. This error may not emerge in situations when students encounter terms that do not involve the distributive property or when students use the distributive property without the element of terms. The confusion seems to be an interaction between students' partial understanding of terms and their partial understanding of the distributive property—which may not be revealed unless both are present. Expiration date: Grade 5 (5.OA.1).

9. Improper fractions should always be written as a mixed number.

When students are first learning about fractions, they are often taught to always change improper fractions to mixed numbers, perhaps so they can better visualize how many *wholes* and *parts* the number represents. This rule can certainly help students understand that positive mixed numbers can represent a value greater than one whole, but it can be troublesome when students are working within a specific mathematical context or real-world situation that requires them to use improper fractions. This frequently first occurs when students begin using improper fractions to compute and again when students later learn about the slope of a line and must represent the slope as the rise/run, which is sometimes appropriately and usefully expressed as an improper fraction. Expiration dates: Grade 5 (5.NF.1) and again in Grade 7 (7.RP.2).

10. The number you say first in counting is always less than the number that comes next.

In the early development of number, students are regularly encouraged to think that number

TABLE 1

Some commonly used language "expires " and should be replaced with more appropriate alternatives.

Expired mathematical language and suggested alternatives

What is stated	What should be stated
Using the words *borrowing or carrying* when subtracting or adding, respectively	Use *trading* or *regrouping* to indicate the actual action of trading or exchanging one place value unit for another unit.
Using the phrase ___ *out of* __ to describe a fraction, for example, one out of seven to describe $\frac{1}{7}$	Use the fraction and the attribute. For example, say *one-seventh of the length of the string.* The *out of* language often causes students to think a part is being subtracted from the whole amount (Philipp, Cabral, and Schappelle 2005).
Using the phrase *reducing fractions*	Use *simplifying fractions*. The language of *reducing* gives students the incorrect impression that the fraction is getting smaller or being reduced in size.
Asking how shapes are *similar* when children are comparing a set of shapes	Ask, *How are these shapes the same? How are the shapes different?* Using the word *similar* in these situations can eventually confuse students about the mathematical meaning of similar, which will be introduced in middle school and relates to geometric figures.
Reading the equal sign as *makes*, for example, saying, *Two plus two makes four* for 2 + 2 = 4	Read the equation 2 + 2 = 4 as *Two plus two equals or is the same as four.* The language *makes* encourages the misconception that the equal sign is an action or an operation rather than representative of a relationship.
Indicating that a number *divides evenly* into another number	Say that a number *divides* another number a whole number of times or that it *divides without a remainder.*
Plugging a number into an expression or equation	Use *substitute values* for an unknown.
Using *top number* and *bottom number* to describe the numerator and denominator of a fraction, respectively	Students should see a fraction as one number, not two separate numbers. Use the words *numerator* and *denominator* when discussing the different parts of a fraction.

relationships are fixed. For example, the relationship between 3 and 8 is always the same. To determine the relationship between two numbers, the numbers must implicitly represent a count made by using the same unit. But when units are different, these relationships change. For example, three dozen eggs is more than eight eggs, and three feet is more than eight inches. Expiration date: Grade 2 (2.MD.2).

11. The longer the number, the larger the number.

The length of a number, when working with whole numbers that differ in the number of digits, does indicate this relationship or magnitude. However, it is particularly troublesome to apply this rule to decimals (e.g., thinking that 0.273 is larger than 0.6), a misconception noted by Desmet, Grégoire, and Mussolin (2010). Expiration date: Grade 4 (4.NF.7).

12. Please Excuse My Dear Aunt Sally.

This phrase is typically taught when students begin solving numerical expressions involving multiple operations, with this mnemonic serving as a way of remembering the order of operations. Three issues arise with the application of this rule. First, students incorrectly believe that they should always do multiplication before division, and addition before subtraction, because of the order in which they appear in the mnemonic PEMDAS (Linchevski and Livneh 1999). Second, the order is not as strict as students are led to believe. For example, in the expression $3^2 – 4(2 + 7) + 8 \div 4$, students have options as to where they might start. In this case, they may first simplify the 2 + 7 in the grouping symbol, simplify 3^2, or divide before doing any other computation—all without affecting the outcome. Third, the *P* in PEMDAS suggests that parentheses are first, rather than

Other rules that expire

We invite *Teaching Children Mathematics* (*TCM*) readers to submit additional instances of "rules that expire" or "expired language" that this article does not address. If you would like to share an example, please use the format of the article, stating the rule to avoid, a case of how it expires, and when it expires in the Common Core State Standards for Mathematics. If you submit an illustration of expired language, include "What is stated" and "What should be stated" (see **table 1**).

Join us as we continue this conversation on *TCM*'s blog at **www.nctm.org/TCMblog/MathTasks** or send your suggestions and thoughts to **tcm@nctm.org**. We look forward to your input.

grouping symbols more generally, which would include brackets, braces, square root symbols, and the horizontal fraction bar. Expiration date: Grade 6 (6.EE.2).

13. The equal sign means *Find the answer* or *Write the answer*.

An equal sign is a relational symbol. It indicates that the two quantities on either side of it represent the same amount. It is not a signal prompting the answer through an announcement to "do something" (Falkner, Levi, and Carpenter 1999; Kieran 1981). In an equation, students may see an equal sign that expresses the relationship but cannot be interpreted as *Find the answer*. For example, in the equations below, the equal sign provides no indication of an answer. Expiration date: Grade 1 (1.OA.7).

$$6 = \square + 4$$
$$3 + x = 5 + 2x$$

Expired language

In addition to helping students avoid the thirteen rules that expire, we must also pay close attention to the mathematical language we use as teachers and that we allow our students to use. The language we use to discuss mathematics (see **table 1**) may carry with it connotations that result in misconceptions or misuses by students, many of which relate to the Thirteen Rules That Expire listed above. Using accurate and precise vocabulary (which aligns closely with SMP 6) is an important part of developing student understanding that supports student learning and withstands the need for complexity as students progress through the grades.

No expiration date

One characteristic of the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) is to have fewer, but deeper, more rigorous standards at each grade—and to have less overlap and greater coherence as students progress from K–grade 12. We feel that by using consistent, accurate rules and precise vocabulary in the elementary grades, teachers can play a key role in building coherence as students move from into the middle grades and beyond. No one wants students to realize in the upper elementary grades or in middle school that their teachers taught "rules" that do not hold true.

With the implementation of CCSSM, now is an ideal time to highlight common instructional practices that teachers can tweak to better prepare students and allow them to have smoother transitions moving from grade to grade. Additionally, with the implementation of CCSSM, many teachers—even those teaching the same grade as they had previously—are being required to teach mathematics content that differs from what they taught in the past. As teachers are planning how to teach according to new standards, now is a critical point to think about the rules that should or should not be taught and the vocabulary that should or should not be used in an effort to teach in ways that do not "expire."

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